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Irreducible representations of Brauer algebras

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Abstract. Irreducible representations of Brauer algebras are constructed by using the induced representation and the linear equation method. As examples, some matrix representations of Brauer algebras $D_f(n)$ with $f \leqslant 5$ are presented.

1. Introduction

Brauer algebras [1, 2] $D_f(n)$, which are similar to the group algebra of the symmetric group S_f related to the decomposition of f-rank tensors of the general linear group $GL(n)$, are the centralizer algebras of the orthogonal group $O(n)$ or the symplectic group $Sp(2m)$ when $n = -2m$. Using the complementary relation or the so-called Schur–Weyl duality relation between S_f and $U(n)$, one can obtain the knowledge of the representation theory of $U(n)$, such as basis vectors, coupling and recoupling coefficients from the symmetric group S_f [5–8]. The Brauer algebras $D_f(n)$ play a similar role for other classical Lie groups. More precisely, if G is the orthogonal group $O(n)$ or the sympletic group $Sp(2m)$, the corresponding centralizer algebra $B_f(G)$ are quotients of Brauer's $D_f(n)$ and $D_f(-2m)$, respectively [2, 4].

On the other hand, the Brauer algebras $D_f(n)$ are a special case of Birman–Wenzl algebras [3]. The Birman–Wenzl algebras $C_f(q, r)$ appear in connection with the Kauffman link invariant and quantum groups of types B, C, D [4]. The Birman–Wenzl algebras $C_f(q, r)$ are a special realization of braid group. Unitary braid representations play an important role in the study of subfactors and in quantum field theory [15, 16]. If the parameters q and r are not roots of unity, representations of $C_f(q, r)$ vary continuously with q and r. Thus one can obtain the information about the representations of $C_f(q, r)$ from those of $D_f(n)$ for $n \geqslant f - 1$ or non-integer n.

In this paper, we will outline a method for constructing irreducible representations of $D_f(n)$. In section 2, we will briefly review the definitions and some important properties of $D_f(n)$. In section 3, we will outline an induced representation method for constructing irreps of $D_f(n)$. As examples, some orthogonal matrix representations of $D_f(n)$ will be derived by using the linear equation method (LEM) [6–8]. The results will be presented in section 4. The technique developed in this paper can also be extended to the Birman–Wenzl algebra $C_f(q, r)$ case by using the results of Hecke algebra representations proposed previously [6–8].

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2. The algebras $D_f(n)$

$D_f(n)$ can be defined algebraically by $2f-2$ generators $\{g_1, g_2, \ldots, g_{f-1}, e_1, e_2, \ldots, e_{f-1}\}$, which satisfy the following relations:

$$g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1} \tag{2.1a}$$

$$g_i g_j = g_j g_i \qquad \text{for } |i-j| \geqslant 2 \tag{2.1b}$$

$$e_i g_i = e_i \tag{2.1c}$$

$$e_i g_{i-1} e_i = e_i \tag{2.1d}$$

$$e_i^2 = n e_i \tag{2.1e}$$

$$(g_i - 1)^2 (g_i + 1) = 0. \tag{2.1f}$$

Using the above-defined relations or by drawing pictures of link diagrams (cf [4]), we can obtain the following relations which are useful for our purposes:

$$e_{i\pm1} g_i g_{i\pm1} = g_i g_{i\pm1} e_i \tag{2.2a}$$

$$g_i e_{i\pm1} g_i = g_{i\pm1} e_i g_{i\pm1} \tag{2.2b}$$

$$e_i g_{i\pm1} e_i = e_i \tag{2.2c}$$

$$g_i e_{i\pm1} e_i = g_{i\pm1} e_i \tag{2.2d}$$

$$e_i e_j = e_j e_i \qquad \text{for } |i-j| \geqslant 2 \tag{2.2e}$$

$$e_i e_{i\pm1} e_i = e_i. \tag{2.2f}$$

It is clear that $\{g_1, g_2, \ldots, g_{f-1}\}$ generate a subalgebra S_f, i.e. $D_f(n) \supset S_f$.

The properties of $D_f(n)$ have been discussed by many authors [2–4, 10–13]. Based on their results, it is known that $D_f(n)$ is semisimple, i.e. it is a direct sum of full matrix algebras over $\mathbb{C}$, when n is not an integer or is an integer with $n \geqslant f-1$, otherwise $D_f(n)$ is no longer semisimple. Whenever $D_f(n)$ is semisimple, its irreducible representations can be labelled by a Young diagram with f, $f-2$, $f-4$, …, 1 or 0 boxes. It can be seen that by removing the generators e_{f-1} and g_{f-1}, $\{g_1, g_2, \ldots, g_{f-2}, e_1, e_2, \ldots, e_{f-2}\}$ generate $D_{f-1}(n)$. By doing so repeatedly, one can establish the standard algebraic chain $D_f(n) \supset D_{f-1}(n) \supset \cdots \supset D_2(n)$. We call it the standard basis of $D_f(n)$. Let Γ_f be the set of all Young diagrams with $k \leqslant f$ boxes such that $k \geqslant 0$ and $f-k$ is even. As was pointed out in [2] and [4], if the algebra $D_f(n)$ is semisimple, it decomposes into a direct sum of the full matrix algebras $D_{f,[\lambda]}(n)$, where $[\lambda] \in \Gamma_f$. If $V_{f,[\lambda]}$ is a simple $D_{f,[\lambda]}(n)$ module, it decomposes as a $D_{f-1}(n)$ module into a direct sum

$$V_{f,\lambda} = \oplus_{[\mu] \leftrightarrow [\lambda]} V_{f-1,[\mu]}$$

where $V_{f-1,[\mu]}$ is a simple $D_{f-1,[\mu]}(n)$ module and $[\mu]$ runs through all diagrams obtained by removing or (if $[\lambda]$ contains less than f boxes) adding a box to $[\lambda]$. In what follows, we always assume that $D_f(n)$ is semisimple.

3. Construction of basis vectors for irreducible representations of $D_f(n)$

As in the symmetric group S_f case, in order to label the standard basis of $D_f(n)$, we need a set of indices $\{1, 2, \ldots, f\}$. Firstly, k-time trace contraction basis vectors can be denoted by

$$\left|(\overbrace{1\,2}\;\overbrace{3\,4}\cdots\overbrace{2k-1\;2k})(\omega_0)=(2k+1,2k+2,\ldots,f)\right\rangle\equiv e_1e_3\cdots e_{2k-1}|(123\cdots f)\rangle\,. \tag{3.1}$$

Then, any normal ordered basis vectors [8] can be written as

$$\begin{aligned}&\left|(\overbrace{a_1a_2}\overbrace{a_3a_4}\cdots\overbrace{a_{2k-1}a_{2k}})(\omega')=(a_{2k+1},a_{2k+2},\ldots,a_f)\right\rangle\\&\quad=Q_\omega\left|(\overbrace{1\,2}\;\overbrace{3\,4}\cdots\overbrace{2k-1\;2k})(\omega_0)\right\rangle\end{aligned} \tag{3.2}$$

where $a_1<a_2,\ a_3<a_4,\ldots,\ a_{2k-1}<a_{2k};\ a_{2k+1}<a_{2k+2}<\cdots<a_f$, and Q_ω is the so-called order preserving permutation operators, which are also the left coset representatives in the decomposition

$$S_f=\sum_\omega\oplus Q_\omega((S_2\otimes)^kS_{f-2k}). \tag{3.3}$$

For example, when $f=3$ and $k=1$, we have $Q_\omega=\{1,g_1,g_1g_2\}$. The ordering of the sequences (ω) is specified in the following way. We regard the part $(\omega_1)=(a_1,a_2)$ in $\{(a_1a_2a_3a_4\cdots a_{2k-1}a_{2k})\ (\omega')\}$ as a vector of length 2. If the last non-zero component of the vector $(\omega_1)-(\bar{\omega}_1)$ is less than zero, then we say $(\omega)<(\bar{\omega})$. This ordering of (ω) is consistent with that for symmetric groups [5]. In fact, $2k$ indices in (3.2) are contracted. The remaining $f-2k$ indices $\{a_{2k+1},a_{2k+2},\ldots,a_f\}$ can be assigned to a permutation symmetry $[\lambda]$, a Young diagram with $f-2k$ boxes, with respect to the S_{f-2k} action. Hence, for any irreducible representation of $S_{f-2k}(\omega')$, we can use orthogonal vectors $\{|Y_m^{[\lambda]}(\omega')\rangle\}$ to label the standard basis vectors of S_{f-2k}, where $Y_m^{[\lambda]}$ is a standard Young tableau, (ω') is a set of indices filled in $Y_m^{[\lambda]}$, and m can be understood either as the Yamanouchi symbols or the indices of the basis vectors in the so-called decreasing page order of the Yamanouchi symbols [5].

As was proved in [9], the space $V_k^{[\lambda]}$ spanned by

$$\left\{Q_\omega\left|(\overbrace{1\,2}\;\overbrace{3\,4}\cdots\overbrace{2k-1\;2k})Y_m^{[\lambda]}(\omega_0)\right\rangle\right\}$$

is $D_f(n)$ irreducible. This can be proved by direct computation with the help of (2.1) and (2.2). Hence, the basis vectors of $D_f(n)$ irrep $[\lambda]$ with $f-2k$ boxes can be expressed in terms of a linear combination of the basis vectors in $V_k^{[\lambda]}$. In fact, what we have constructed is the $(D_2(n)\otimes)^kD_{f-2k}(n)\uparrow D_f(n)$ induced representation for the outer product $([0]\otimes)^k[\lambda]\uparrow[\lambda]$. $V_k^{[\lambda]}$ is quite simply the space spanned by the uncoupled normal ordered basis vectors of $(D_2(n)\otimes)^kD_{f-2k}(n)$.

As was pointed out in [2], the dimensions of irreducible representations of $D_f(n)$ can be computed by using Bratteli diagrams inductively. Using combinatorial method to compute the different ways of k-time trace contraction among f indices, we can prove that the dimension formula for irreps of $D_f(n)$ can be expressed [9] as

$$\dim(D_f(n);[\lambda]_{f-2k})=\frac{f!}{(f-2k)!(2k)!!}\dim(S_{f-2k};[\lambda]) \tag{3.4}$$

where $[\lambda]_{f-2k}$ denotes a Young diagram with $f-2k$ boxes, and $\dim(S_{f-2k};[\lambda])$ is the dimension for the irrep $[\lambda]$ of S_{f-2k}, which can further be expressed, for example, by Robinson's formula for irreps of symmetric groups.

It should be noted that the labelling scheme and the decomposition for $D_f(n)$ are the same as those for Birman–Wenzl algebras $C_f(q,r)$ when q and r are not roots of unity.

Thus the dimension formula (3.4) also applies to Birman–Wenzl albgebras $C_f(q,r)$ when q and r are not roots of unity.

As was mentioned earlier, $D_f(n)$ contains S_f as a subalgebra. Hence an irrep $[\lambda]$ of S_f is also the same irrep of $D_f(n)$, in which one simply takes that $e_i = 0$ for $i = 1, 2, \ldots, f-1$. i.e. there is no trace contraction in such a representation. So we only need to discuss the irreps $[\lambda]_{f-2k}$ of $D_f(n)$ for $k \neq 0$. For $D_2(n)$, there are trivially 3 one-dimensional irreps [0], [2], and $[1^2]$ with

$$\begin{aligned} g_1\,|[0]\rangle &= |[0]\rangle & e_1\,|[0]\rangle &= n\,|[0]\rangle \\ g_1\,|[2]\rangle &= |[2]\rangle & e_1\,|[2]\rangle &= 0 \\ g_1|[1^2]\rangle &= -|[1^2]\rangle & e_1|[1^2]\rangle &= 0. \end{aligned} \tag{3.5}$$

The non-trivial cases occur when $f \geqslant 3$. In what follows, we will restrict ourselves to integer n with $n \geqslant f-1$. The results for non-integer n and negative n values can be obtained by using n-continuation and algebraic isomorphic maps, i.e. the results are also valid for any permitted n values. This will be discussed later.

When n is a positive integer, we can use tensor products of the rank-1 unit tensor operator of $O(n)$ to construct the basis of $D_f(n)$ in the standard basis explicitly. In this case the indices $1, 2, \ldots, f$ are used to distinguish tensor operators from different spaces. We also need a set of the corresponding indices $i_1, i_2, \ldots, i_f$ to label the tensor components which can be taken as n different values, namely

$$T^1_{i_1} T^2_{i_2} \cdots T^f_{i_f} \equiv T^{12\cdots f}_{i_1 i_2 \cdots i_f}. \tag{3.6}$$

The actions of g_i and e_i on (3.6) are given by

$$g_i T^{12\cdots i\ i+1\cdots f}_{j_1 j_2 \cdots j_i j_{i+1} \cdots j_f} = T^{12\cdots i+1\ i\cdots f}_{j_1 j_2 \cdots j_i j_{i+1} \cdots j_f} \tag{3.7}$$

$$e_i T^{12\cdots i\ i+1\cdots f}_{j_1 j_2 \cdots j_i j_{i+1} \cdots j_f} = \delta_{j_i j_{i+1}} T^{12\cdots i\ i+1\cdots f}_{j_1 j_2 \cdots j_i j_{i+1} \cdots j_f} \tag{3.8}$$

i.e. the generator $\{g_i\}$ is a permutation of tensors in ith and $(i+1)$th spaces, while e_i is a trace contraction of the corresponding tensor components. We assume that $\{T^{12\cdots f}_{j_1 j_2 \ldots j_f}\}$ spans a orthonormal inner product space, namely

$$\left(T^{1'2'\cdots f'}_{j'_1 j'_2 \cdots j'_f}, T^{12\cdots f}_{j_1 j_2 \cdots j_f}\right) = \prod \delta_{ii'} \delta_{j_i j'_i}. \tag{3.9}$$

The star operation, a conjugate linear map $\dagger$, on $D_f(n)$ is defined [4] by

$$g_i^\dagger = g_i \qquad i = 1, 2, \ldots, f-1 \tag{3.10a}$$

$$e_i^\dagger = e_i \qquad i = 1, 2, \ldots, f-1 \tag{3.10b}$$

which are neccessary in deriving the matrix representations of $D_f(n)$. Because of the contraction, the uncoupled normal ordered basis vectors given by (3.2) are no longer orthonormal. For example, when $f = 3$, and $k = 1$, we have

$$
\begin{aligned}
\langle \overbrace{1\,2}\ 3 | \overbrace{1\,2}\ 3 \rangle &= \left(T_{iij}^{123}, T_{iij}^{123}\right) = n \\
\langle \overbrace{1\,3}\ 2 | \overbrace{1\,3}\ 2 \rangle &= \left(T_{iij}^{132}, T_{iij}^{132}\right) = n \\
\langle \overbrace{2\,3}\ 1 | \overbrace{2\,3}\ 1 \rangle &= \left(T_{iij}^{231}, T_{iij}^{231}\right) = n \\
\langle \overbrace{1\,2}\ 3 | \overbrace{2\,3}\ 1 \rangle &= \left(T_{iij}^{123}, T_{iij}^{231}\right) = 1 \\
\langle \overbrace{1\,2}\ 3 | \overbrace{1\,3}\ 2 \rangle &= \left(T_{iij}^{123}, T_{iij}^{132}\right) = 1 \\
\langle \overbrace{1\,3}\ 2 | \overbrace{2\,3}\ 1 \rangle &= \left(T_{iij}^{132}, T_{iij}^{231}\right) = 1.
\end{aligned}
\tag{3.11}
$$

If we relabel the above basis vectors by

$$
|1\rangle = |\overbrace{1\,2}\ 3\rangle \qquad |2\rangle = |\overbrace{1\,3}\ 2\rangle \qquad |3\rangle = |\overbrace{2\,3}\ 1\rangle
$$

the norm matrix with elements $\langle i|j\rangle = \langle j|i\rangle$ for $1 \leqslant i, j \leqslant 3$ is just the matrix of $T_{2,1}$ defined by Hanlon and Wales [10].

In what follows, we will use the induced representation $(D_2(n)\otimes)^k D_{f-2k}(n) \uparrow D_f(n)$ for the outer product $([0]\otimes)^k[\lambda]) \uparrow [\lambda]$ to derive the irreducible representations of $D_f(n)$. The basis vectors of $[\lambda]_{f-2k}$ is denoted by

$$
\left| \begin{matrix} [\lambda]_{f-2k} & D_f(n) \\ [\mu] & D_{f-1}(n) \\ \vdots & \vdots \\ [\rho] & D_{f-p+1}(n) \\ [\nu] & D_{f-p}(n) \\ \vdots & \vdots \end{matrix} \right) = \left| \begin{matrix} [\lambda]_{f-2k} \\ [\mu] \\ \vdots \\ [\rho] \\ Y_M^{[\nu]}(\bar{\omega}) \end{matrix} \right)
\tag{3.12}
$$

where $(\bar{\omega}) = (1, 2, \ldots, f - p)$, and $[\mu]$ can be taken as a Young diagram obtained by removing or (if $[\lambda]$ contains less than f boxes) adding a box to $[\lambda]$. By repeatedly doing so to p steps, there always exists a Young diagram $[\nu]$ with $f - p$ boxes which corresponds to an irrep of $D_{f-p}(n)$. Thus $[\nu]$ is identical to the same irrep of S_{f-p}. So the labelling scheme of the remaining part can be assigned to a standard Young tableau $Y_M^{[\nu]}$ with $f - p$ indices $\{1, 2, \ldots, f - p\}$.

For example, for the irrep $[1]$ of $D_3(n)$, we have three basis vectors

$$
\left| \begin{matrix} [1] \\ 0 \end{matrix} \right) \qquad \left| \begin{matrix} [1] \\ \boxed{1}\boxed{2} \end{matrix} \right) \qquad \left| \begin{matrix} [1] \\ \boxed{1} \\ \boxed{2} \end{matrix} \right)
\tag{3.13}
$$

under the standard basis $D_3(n) \supset D_2(n)$.

We will now use the linear equation method [8] to derive the irreducible representations of $D_f(n)$ inductively. Firstly, the results of generators $\{g_1, g_2, \ldots, g_{f-1}, e_1, e_2, \ldots, e_{f-1}\}$ acting on (3.2) can be found directly by using the algebraic relations given by (2.1) and (2.2), and the standard results of the symmetric groups which are required when both i and $i + 1$ are in the Young tableau $Y_m^{[\lambda]}$. Secondly, we assume the matrix representations of

$D_{f-1}(n) \supset D_{f-2}(n) \supset \cdots \supset D_2(n)$ are completely known. The basis vectors for $[\lambda]_{f-2k}$ of $D_f(n)$ can be expressed as

$$\left|\begin{matrix} [\lambda]_{f-2k} & D_f(n) \\ [\mu] & D_{f-1}(n) \\ \vdots & \vdots \\ [\rho] & D_{f-p+1}(n) \\ Y_M^{[\nu]} & D_{f-p}(n) \end{matrix}\right\rangle = \sum_{\omega,m} C_{[0],[\lambda]m,\omega}^{[\lambda]_{f-2k}\rho[\nu]M} Q_\omega \left|(\widehat{1\,2}\ \widehat{3\,4}\cdots\widehat{2k-1\ 2k});\ Y_m^{[\lambda]}(\omega_0)\right\rangle \tag{3.14}$$

where the $C_{[0],[\lambda]m,\omega}^{[\lambda]_{f-2k}\rho[\nu]M}$ are the induction coefficients (IDCs) of $(D_2(n)\otimes)^k D_{f-2k}(n) \uparrow D_f(n)$ for the outer product $([0]\otimes)^k[\lambda] \uparrow [\lambda]$, which need to be determined, and $\rho \equiv [\mu]\cdots[\rho]$. The IDCs satisfy the following orthogonality relations:

$$\sum_{\omega,m,\omega',m'} C_{[0],[\lambda]m',\omega'}^{[\lambda]_{f-2k}\rho'[\nu']M'} C_{[0],[\lambda]m,\omega}^{[\lambda]_{f-2k}\rho[\nu]M} \left\langle(\widehat{a_1a_2}\cdots\widehat{a_{2k-1}a_{2k}}),\ Y_m^{[\lambda]}(\omega)\middle|(\widehat{a_1'a_2'}\cdots\widehat{a_{2k-1}'a_{2k}'})Y_{m'}^{[\lambda]}(\omega')\right\rangle$$
$$= \delta_{\rho\rho'}\delta_{[\nu][\nu']}\delta_{MM'} \tag{3.15}$$

where we have assumed that the basis vectors of $D_f(n) \supset D_{f-1}(n) \supset \cdots \supset D_2(n)$ given by (3.12) are orthonormal, i.e.

$$\left(\begin{matrix} [\lambda] \\ [\mu] \\ \vdots \\ [\rho] \\ Y_M^{[\nu]} \end{matrix}\middle|\begin{matrix} [\lambda] \\ [\mu'] \\ \vdots \\ [\rho'] \\ Y_{M'}^{[\nu']} \end{matrix}\right) = \delta_{\mu\mu'}\delta_{\rho\rho'}\cdots\delta_{\nu\nu'}\delta_{MM'}. \tag{3.16}$$

This coincides with the results of symmetric groups when the irrep $[\lambda]$ of $D_f(n)$ has f boxes.

Applying the operators R_i $(= g_i$ or $e_i)$ with $i = 1, 2, \ldots, f-2$ to (3.14), the left-hand side of (3.14) becomes

$$\sum_{\omega m\rho'[\nu']M'} C_{[0],[\lambda]m,\omega}^{[\lambda]\rho'[\nu']M'} \left\langle[\lambda]\rho[\nu]M\middle|R_i\middle|[\lambda]\rho'[\nu']M'\right\rangle Q_\omega\left|(\widehat{1\,2}\ \widehat{3\,4}\cdots\widehat{2k-1\ 2k})Y_{M'}^{[\nu']}(\omega_0)\right\rangle. \tag{3.17}$$

While the right-hand side of (3.14) becomes

$$\sum_{\omega m} C_{[0],[\lambda]m,\omega}^{[\lambda]\rho[\nu]M}(R_iQ_\omega)\left|(\widehat{1\,2}\ \widehat{3\,4}\cdots\widehat{2k-1\ 2k})Y_m^{[\nu]}(\omega_0)\right\rangle. \tag{3.18}$$

Combining (3.17) and (3.18), we get

$$\sum_{\rho'[\nu']M'} C_{[0],[\lambda]m,\omega}^{[\lambda]\rho'[\nu']M'} \langle[\lambda]\rho[\nu]M|\,R_i\,\left|[\lambda]\rho'[\nu']M'\right\rangle = C_{[0],[\lambda]m',\omega'}^{[\lambda]\rho[\nu]M} f_i \tag{3.19}$$

where $C_{[0],[\lambda]m',\omega'}^{[\lambda]\rho[\nu]M} f_i$ is the coefficient in front of

$$Q_\omega\left|(\widehat{1\,2}\ \widehat{3\,4}\cdots\widehat{2k-1\ 2k})Y_m^{[\nu]}(\omega_0)\right\rangle$$

after applying R_i to the right-hand side of (3.14).

The number of independent basis vectors given by (3.14) and those by (3.2) all equal to $\dim([\lambda]_{f-2k}; D_f(n))$ given by (3.4). For a given irrep $[\lambda]_{f-2k}$, there are $[\dim([\lambda]_{f-2k};$

$D_f(n))]^2$ IDCs. Equation (3.19) will yield $2(f-1)[\dim([\lambda]_{f-2k};\ D_f(n))]^2$ linear relations among the IDCs. As in the Hecke algebra $H_n(q)$ case [8], there are many redundant relations among IDCs. However, using equation (3.19) together with the orthogonality relations (3.15), we can establish $[\dim([\lambda]_{f-2k};\ D_f(n))]^2$ sufficient relations among these IDCs, which can be used to solve them.

In the calculation, the relative phase of the IDCs is determined completely by (3.15), and (3.19), while the overall phase is fixed by requiring that the IDCs with $m = M = 1$ and with smallest possible index ω be positive

$$C^{[\lambda]\rho[\nu]M=1}_{[0],[\lambda]m=1,\omega=\min} > 0. \tag{3.20}$$

This phase convention is consistent with that for symmetric groups [5].

Once these IDCs are known, the orthonormal basis vectors given by (3.14) are completely determined. The matrix representations of R_{f-1} ($= g_{f-1}$ or e_{f-1}) can then be derived by directly acting R_{f-1} on (3.14) with the known matrix elements of R_{f-1} under the uncoupled normal ordered basis (3.2). Using this method and starting from the results given by (3.5), one can obtain the matrix representations of $D_f(n)$ under the standard basis. In what follows, we will give an example to show how this method works.

Example. Deriving matrix representations of $D_3(n)$. The irreducible representations of $D_3(n)$ with three boxes are the same as those of S_3. We only need to derive the three-dimensional irrep [1]. The process consists of the following steps:

Step 1. Write the basis vectors of $D_3(n)$ in terms of uncoupled normal ordered basis vectors with 1-contraction, and calculate the norm matrix elements

$$\left|\begin{matrix}[1]\\0\end{matrix}\right\rangle = \sum_{i=1}^{3} a_i\,|i\rangle \qquad \left|\begin{matrix}[1]\\ \begin{array}{|c|c|}\hline 1 & 2\\\hline\end{array}\end{matrix}\right\rangle = \sum_{i=1}^{3} b_i\,|i\rangle \qquad \left|\begin{matrix}[1]\\ \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}\end{matrix}\right\rangle = \sum_{i=1}^{3} c_i\,|i\rangle \tag{3.21}$$

where $|i\rangle$ $(i = 1,\ 2,\ 3)$ are given after (3.11), and a_i, b_i and c_i are the corresponding IDCs. The norm matrix with elements $\langle i|j\rangle$ for $1 \leqslant i, j \leqslant 3$ is

$$\begin{pmatrix} n & 1 & 1\\ 1 & n & 1\\ 1 & 1 & n\end{pmatrix}. \tag{3.22}$$

Step 2. Derive the linear relations among the IDCs. Applying generators g_1 and e_1, respectively to (3.21), we obtain

$$a_1 \neq 0 \qquad a_2 = 0 \qquad a_3 = 0 \tag{3.23a}$$

$$b_1 \neq 0 \qquad b_2 = b_3 = -\frac{n}{2}b_1 \tag{3.23b}$$

$$c_1 = 0 \qquad c_2 = -c_3. \tag{3.23c}$$

Thus, we have

$$\begin{aligned}
\left|\begin{matrix}[1]\\0\end{matrix}\right\rangle &= (-)^{\delta_a}a_1\,|1\rangle\\
\left|\begin{matrix}[1]\\ \begin{array}{|c|c|}\hline 1 & 2\\\hline\end{array}\end{matrix}\right\rangle &= (-)^{\delta_b}\frac{n}{2}b_1\left(-\frac{2}{n}\,|1\rangle + |2\rangle + |3\rangle\right)\\
\left|\begin{matrix}[1]\\ \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}\end{matrix}\right\rangle &= (-)^{\delta_c}c_1(|3\rangle - |2\rangle)
\end{aligned} \tag{3.24}$$

where $\delta_a = 0$, $\delta_b = 1$ and $\delta_c = 1$ according to our phase convention. The norm factors a_1, b_1, and c_1 can easily be derived by using the norm matrix obtained in step 1.

$$a_1 = \sqrt{\frac{1}{n}} \qquad b_1 = \sqrt{\frac{2}{n(n+2)(n-1)}} \qquad c_1 = \sqrt{\frac{1}{2(n-1)}}. \tag{3.25}$$

Step 3. Derive the matrix representations of g_2 and e_2 under the standard basis of $D_3(n)$. Applying g_2 to (3.24) and using the relations given by (2.1) and (2.2), one has

$$\begin{aligned}
g_2\left|\begin{matrix}[1]\\0\end{matrix}\right\rangle &= \sqrt{\frac{1}{n}}g_2\,|1\rangle = \frac{1}{n}\left|\begin{matrix}[1]\\0\end{matrix}\right\rangle - \frac{\sqrt{2(n+2)(n-1)}}{2n}\left|\begin{matrix}[1]\\\begin{array}{|c|c|}\hline 1&2\\\hline\end{array}\end{matrix}\right\rangle \\
&\quad + \sqrt{\frac{n-1}{2n}}\left|\begin{matrix}[1]\\\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}\end{matrix}\right\rangle \\
g_2\left|\begin{matrix}[1]\\\begin{array}{|c|c|}\hline 1&2\\\hline\end{array}\end{matrix}\right\rangle &= \sqrt{\frac{n}{2(n+2)(n-1)}}g_2\left(\frac{2}{n}|1\rangle - |2\rangle - |3\rangle\right) \\
&= -\sqrt{\frac{(n+2)(n-1)}{2n^2}}\left|\begin{matrix}[1]\\0\end{matrix}\right\rangle + \frac{n-2}{2n}\left|\begin{matrix}[1]\\\begin{array}{|c|c|}\hline 1&2\\\hline\end{array}\end{matrix}\right\rangle + \sqrt{\frac{n+2}{4n}}\left|\begin{matrix}[1]\\\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}\end{matrix}\right\rangle \\
g_2\left|\begin{matrix}[1]\\\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}\end{matrix}\right\rangle &= \sqrt{\frac{1}{2(n-1)}}g_2\left(|2\rangle - |3\rangle\right) \\
&= \sqrt{\frac{n-1}{2n}}\left|\begin{matrix}[1]\\0\end{matrix}\right\rangle + \sqrt{\frac{n+2}{4n}}\left|\begin{matrix}[1]\\\begin{array}{|c|c|}\hline 1&2\\\hline\end{array}\end{matrix}\right\rangle + \frac{1}{2}\left|\begin{matrix}[1]\\\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}\end{matrix}\right\rangle .
\end{aligned} \tag{3.26a}$$

While applying e_2 to (3.24), one has

$$\begin{aligned}
e_2\left|\begin{matrix}[1]\\0\end{matrix}\right\rangle &= \frac{1}{n}\left|\begin{matrix}[1]\\0\end{matrix}\right\rangle - \sqrt{\frac{(n+2)(n-1)}{2n^2}}\left|\begin{matrix}[1]\\\begin{array}{|c|c|}\hline 1&2\\\hline\end{array}\end{matrix}\right\rangle - \sqrt{\frac{n-1}{2n}}\left|\begin{matrix}[1]\\\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}\end{matrix}\right\rangle \\
e_2\left|\begin{matrix}[1]\\\begin{array}{|c|c|}\hline 1&2\\\hline\end{array}\end{matrix}\right\rangle &= -\sqrt{\frac{(n+2)(n-1)}{2n^2}}\left|\begin{matrix}[1]\\0\end{matrix}\right\rangle + \frac{(n+2)(n-1)}{2n}\left|\begin{matrix}[1]\\\begin{array}{|c|c|}\hline 1&2\\\hline\end{array}\end{matrix}\right\rangle \\
&\quad + \frac{(n-1)}{2}\sqrt{\frac{n+2}{n}}\left|\begin{matrix}[1]\\\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}\end{matrix}\right\rangle \\
e_2\left|\begin{matrix}[1]\\\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}\end{matrix}\right\rangle &= -\sqrt{\frac{n-1}{2n}}\left|\begin{matrix}[1]\\0\end{matrix}\right\rangle + \frac{n-1}{2}\sqrt{\frac{n+2}{n}}\left|\begin{matrix}[1]\\\begin{array}{|c|c|}\hline 1&2\\\hline\end{array}\end{matrix}\right\rangle + \frac{n-1}{2}\left|\begin{matrix}[1]\\\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}\end{matrix}\right\rangle .
\end{aligned} \tag{3.26b}$$

Hence, one obtains the three-dimensional irrep [1] of $D_3(n)$ under the standard basis $D_3(n) \supset D_2(n)$. The results are given in (4.2).

4. Some matrix representations of $D_f(n)$ under the standard basis

In this section, we will list some irreducible matrix representations of $D_f(n)$ under the standard basis, which are derived by using the method outlined in section 3. All the irreps with f boxes, i.e. there is no trace contraction, are omitted here because they are indentical to the symmetric group S_f case with $e_i = 0$ for $i = 1, 2, \ldots, f-1$. The results for $f \leqslant 4$ and two examples with $f = 5$ are presented. However, the dimension of the irreps will increase rapidly with increasing of f. In this case, one can derive the results with the help of a computer running Mathematica. For example, two ten-dimensional irreps of $D_5(n)$ were derived by using this facility.

1. $f = 2$, $[\lambda] = [0]$ with dim = 1. The uncoupled normal ordered basis vector is $|1\rangle = e_1\,|(12)\rangle$ with $\langle 1|1\rangle = n$. The orthonormal basis vector is

$$|[0]\rangle = \sqrt{\frac{1}{n}}\,|1\rangle\,. \tag{4.1a}$$

The matrices of the generators are

$$g_1 = 1 \qquad e_1 = n. \tag{4.1b}$$

2. $f = 3$, $[\lambda] = [1]$ with dim = 3. The uncoupled normal ordered basis vectors, norm matrix, and the orthonormal basis vectors have already been given in section 3. The matrices of the generators g_2 and e_2 are

$$g_2 = \begin{pmatrix} \frac{1}{n} & -\sqrt{\frac{(n+2)(n-1)}{2n^2}} & \sqrt{\frac{n-1}{2n}} \\ -\sqrt{\frac{(n+2)(n-1)}{2n^2}} & \frac{n-2}{2n} & \sqrt{\frac{n+2}{4n}} \\ \sqrt{\frac{n-1}{2n}} & \sqrt{\frac{n+2}{4n}} & \frac{1}{2} \end{pmatrix} \tag{4.2a}$$

$$e_2 = \begin{pmatrix} \frac{1}{n} & -\sqrt{\frac{(n+2)(n-1)}{2n^2}} & -\sqrt{\frac{n-1}{2n}} \\ -\sqrt{\frac{(n+2)(n-1)}{2n^2}} & \frac{(n+2)(n-1)}{2n} & \frac{n-1}{2}\sqrt{\frac{n+2}{n}} \\ -\sqrt{\frac{n-1}{2n}} & \frac{n-1}{2}\sqrt{\frac{n+2}{n}} & \frac{n-1}{2} \end{pmatrix}. \tag{4.2b}$$

3. $f = 4$, $[\lambda] = [0]$ with dim = 3. The uncoupled normal ordered basis vectors are

$$|1\rangle = e_1e_3\,|(1234)\rangle \qquad |2\rangle = g_2\,|1\rangle \qquad |3\rangle = g_1g_2\,|1\rangle\,. \tag{4.3a}$$

The norm matrix is

$$\begin{pmatrix} n^2 & n & n \\ n & n^2 & n \\ n & n & n^2 \end{pmatrix}. \tag{4.3b}$$

The orthonormal basis vectors are

$$\left|\begin{matrix}[0]\\ [1]\\ 0\end{matrix}\right\rangle = \frac{1}{n}\,|1\rangle$$

$$\left|\begin{matrix}[0]\\ [1]\\ \boxed{1\,|\,2}\end{matrix}\right\rangle = \sqrt{\frac{1}{2(n+2)(n-1)}}\left(\frac{2}{n}\,|1\rangle - |2\rangle - |3\rangle\right) \tag{4.3c}$$

$$\left|\begin{array}{c}[0]\\ [1]\\ \begin{array}{|c|}\hline 1\\ \hline 2\\ \hline\end{array}\end{array}\right\rangle = \sqrt{\frac{1}{2n(n-1)}}(|2\rangle - |3\rangle).$$

The matrices of g_3 and e_3 are

$$g_3 = \begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix} \qquad e_3 = \begin{pmatrix} n & & \\ & 0 & \\ & & 0 \end{pmatrix}. \tag{4.3d}$$

4. $f = 4$, $[\lambda] = [2]$ with dim = 6. The uncoupled normal ordered basis vectors are

$$\begin{aligned} &|1\rangle = e_1 \left|(12) \quad \begin{array}{|c|c|}\hline 3 & 4\\ \hline\end{array}\right\rangle \qquad |4\rangle = g_3 g_2 |1\rangle \\ &|2\rangle = g_2 |1\rangle \qquad |5\rangle = g_1 g_3 g_2 |1\rangle \\ &|3\rangle = g_1 g_2 |1\rangle \qquad |6\rangle = g_2 g_1 g_3 g_2 |1\rangle . \end{aligned} \tag{4.4a}$$

The norm matrix is

$$\begin{pmatrix} n & 1 & 1 & 1 & 1 & 0 \\ 1 & n & 1 & 1 & 0 & 1 \\ 1 & 1 & n & 0 & 1 & 1 \\ 1 & 1 & 0 & n & 1 & 1 \\ 1 & 0 & 1 & 1 & n & 1 \\ 0 & 1 & 1 & 1 & 1 & n \end{pmatrix}. \tag{4.4b}$$

The orthonormal basis vectors are

$$\begin{aligned}
&\left|\begin{array}{c}[2]\\ [1]\\ 0\end{array}\right\rangle = \sqrt{\frac{1}{n}}\,|1\rangle \\
&\left|\begin{array}{c}[2]\\ [1]\\ \begin{array}{|c|c|}\hline 1 & 2\\ \hline\end{array}\end{array}\right\rangle = \sqrt{\frac{n}{2(n+2)(n-1)}}\left(\frac{2}{n}\,|1\rangle - |2\rangle - |3\rangle\right) \\
&\left|\begin{array}{c}[2]\\ [1]\\ \begin{array}{|c|}\hline 1\\ \hline 2\\ \hline\end{array}\end{array}\right\rangle = \sqrt{\frac{1}{2(n-1)}}\,(|2\rangle - |3\rangle) \\
&\left|\begin{array}{c}[2]\\ \begin{array}{|c|c|c|}\hline 1 & 2 & 3\\ \hline\end{array}\end{array}\right\rangle = \sqrt{\frac{4}{3n(n+2)(n+4)}}\,\left\{|1\rangle + |2\rangle + |3\rangle - \tfrac{1}{2}(n+2)\,(|4\rangle + |5\rangle + |6\rangle)\right\} \\
&\left|\begin{array}{c}[2]\\ \begin{array}{|c|c|}\hline 1 & 2\\ \hline 3 & \multicolumn{1}{c}{}\\ \cline{1-1}\end{array}\end{array}\right\rangle = \sqrt{\frac{1}{6n(n-1)(n-2)}}\,\{2\,|1\rangle - |2\rangle - |3\rangle - (n-1)\,(|4\rangle + |5\rangle - 2\,|6\rangle)\} \\
&\left|\begin{array}{c}[2]\\ \begin{array}{|c|c|}\hline 1 & 3\\ \hline 2 & \multicolumn{1}{c}{}\\ \cline{1-1}\end{array}\end{array}\right\rangle = \sqrt{\frac{1}{2n(n-1)(n-2)}}\{|2\rangle - |3\rangle - (n-1)(|4\rangle - |5\rangle)\}.
\end{aligned} \tag{4.4c}$$

The matrices of g_3 and e_3 are

$$g_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{n-2}{(n+2)(n-1)} & 0 & \frac{n}{n+2}\sqrt{\frac{2(n+4)}{3(n-1)}} & \frac{n}{n-1}\sqrt{\frac{n-2}{3(n+2)}} & 0 \\ 0 & 0 & \frac{1}{n-1} & 0 & 0 & -\frac{\sqrt{n(n-2)}}{n-1} \\ 0 & \frac{n}{n+2}\sqrt{\frac{2(n+4)}{3(n-1)}} & 0 & \frac{n-2}{3(n+2)} & -\frac{1}{3}\sqrt{\frac{2(n+4)(n-2)}{(n+2)(n-1)}} & 0 \\ 0 & \frac{n}{n-1}\sqrt{\frac{n-2)}{3(n+2)}} & 0 & -\frac{1}{3}\sqrt{\frac{2(n+4)(n-2)}{(n+2)(n-1)}} & \frac{2n-1}{3(n-1)} & 0 \\ 0 & 0 & -\frac{\sqrt{n(n-2)}}{n-1} & 0 & 0 & -\frac{1}{n-1} \end{pmatrix} \tag{4.4d}$$

$$e_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{2n}{(n+2)(n-1)} & 0 & \frac{n}{n+2}\sqrt{\frac{2(n+4)}{3(n-1)}} & -\frac{2n}{n-1}\sqrt{\frac{n-2}{3(n+2)}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{n}{n+2}\sqrt{\frac{2(n+4)}{3(n-1)}} & 0 & \frac{n(n+4)}{3(n+2)} & -\frac{n}{3}\sqrt{\frac{2(n+4)(n-2)}{(n+2)(n-1)}} & 0 \\ 0 & -\frac{2n}{n-1}\sqrt{\frac{n-2)}{3(n+2)}} & 0 & -\frac{n}{3}\sqrt{\frac{2(n+4)(n-2)}{(n+2)(n-1)}} & \frac{2n(n-2)}{3(n-1)} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

5. $f = 4$, $[\lambda] = [11]$ with dim = 6. The uncoupled normal ordered basis vectors are

$$|1\rangle = e_1 \left|(12), \begin{array}{|c|}\hline 3 \\ \hline 4 \\ \hline \end{array}\right\rangle \qquad |4\rangle = g_3 g_2 |1\rangle$$

$$|2\rangle = g_2 |1\rangle \qquad |5\rangle = g_1 g_3 g_2 |1\rangle \tag{4.5a}$$

$$|3\rangle = g_1 g_2 |1\rangle \qquad |6\rangle = g_2 g_1 g_3 g_2 |1\rangle .$$

The norm matrix is

$$\begin{pmatrix} n & 1 & 1 & -1 & -1 & 0 \\ 1 & n & 1 & 1 & 0 & -1 \\ 1 & 1 & n & 0 & 1 & 1 \\ -1 & 1 & 0 & n & 1 & -1 \\ -1 & 0 & 1 & 1 & n & 1 \\ 0 & -1 & 1 & -1 & 1 & n \end{pmatrix}. \tag{4.5b}$$

The orthonormal basis vectors are

$$\left| \begin{matrix} [1^2] \\ [1] \\ 0 \end{matrix} \right\rangle = \sqrt{\frac{1}{n}}\, |1\rangle$$

$$\left| \begin{matrix} [1^2] \\ [1] \\ \begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array} \end{matrix} \right\rangle = \sqrt{\frac{n}{2(n+2)(n-1)}} \left\{ \frac{2}{n} |1\rangle - |2\rangle - |3\rangle \right\}$$

$$\left| \begin{matrix} [1^2] \\ [1] \\ \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array} \end{matrix} \right\rangle = \sqrt{\frac{1}{2(n-1)}} \{|2\rangle - |3\rangle\}$$

$$\left| \begin{array}{c} [1^2] \\ \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \end{array} \right\rangle = \sqrt{\frac{2}{(n-1)(n^2-4)}} \left\{ |1\rangle - \frac{1}{2}(|2\rangle + |3\rangle) + \frac{n-1}{2}(|4\rangle + |5\rangle) \right\} \tag{4.5c}$$

$$\left| \begin{array}{c} [1^2] \\ \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array} \end{array} \right\rangle = \sqrt{\frac{1}{6(n-1)(n^2-4)}} \{3(|2\rangle - |3\rangle) + (n-1)(|5\rangle - |4\rangle) + 2(n-1)\,|6\rangle\}$$

$$\left| \begin{array}{c} [1^2] \\ \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array} \end{array} \right\rangle = \sqrt{\frac{1}{3(n-2)}} \{|4\rangle - |5\rangle + |6\rangle\}.$$

The matrices of g_3 and e_3 are

$$g_3 = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{n-1} & 0 & -\frac{\sqrt{n(n-2)}}{n-1} & 0 & 0 \\ 0 & 0 & \frac{1}{n-1} & 0 & -\frac{1}{n-1}\sqrt{\frac{n^2-4}{3}} & \sqrt{\frac{2(n-2)}{3(n-1)}} & 0 \\ 0 & -\frac{\sqrt{n(n-2)}}{n-1} & 0 & -\frac{1}{n-1} & 0 & 0 \\ 0 & 0 & -\frac{1}{n-1}\sqrt{\frac{n^2-4}{3}} & 0 & \frac{2n-5}{3(n-1)} & \frac{1}{3}\sqrt{\frac{2(n+2)}{n-1}} \\ 0 & 0 & \sqrt{\frac{2(n-2)}{3(n-1)}} & 0 & \frac{1}{3}\sqrt{\frac{2(n+2)}{n-1}} & \frac{1}{3} \end{pmatrix} \tag{4.5d}$$

$$e_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{2}{n-1} & 0 & -\frac{2}{n-1}\sqrt{\frac{n^2-4}{3}} & -\sqrt{\frac{2(n-2)}{3(n-1)}} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{2}{n-1}\sqrt{\frac{n^2-4}{3}} & 0 & \frac{2(n^2-4)}{3(n-1)} & \frac{n-2}{3}\sqrt{\frac{2(n+2)}{n-1}} \\ 0 & 0 & -\sqrt{\frac{2(n-2)}{3(n-1)}} & 0 & \frac{n-2}{3}\sqrt{\frac{2(n+2)}{n-1}} & \frac{n-2}{3} \end{pmatrix}.$$

6. $f = 5$, $[\lambda] = [3]$ with dim = 10. The uncoupled normal ordered basis vectors are

$$\begin{aligned} |1\rangle &= e_1 \left|(12), \begin{array}{|c|c|c|}\hline 3 & 4 & 5 \\ \hline\end{array}\right\rangle & |6\rangle &= g_2g_3g_1g_2\,|1\rangle \\ |2\rangle &= g_2\,|1\rangle & |7\rangle &= g_4g_3g_2\,|1\rangle \\ |3\rangle &= g_1g_2\,|1\rangle & |8\rangle &= g_4g_3g_1g_2\,|1\rangle \\ |4\rangle &= g_3g_2\,|1\rangle & |9\rangle &= g_2g_4g_3g_1g_2\,|1\rangle \\ |5\rangle &= g_3g_1g_2\,|1\rangle & |10\rangle &= g_3g_2g_4g_3g_1g_2\,|1\rangle\,. \end{aligned} \tag{4.6a}$$

The norm matrix is

$$\begin{pmatrix} n & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & n & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & n & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & n & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & n & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & n & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & n & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & n & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & n & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & n \end{pmatrix}. \tag{4.6b}$$

The orthonormal basis vectors are

$$\left| \begin{matrix} [3] \\ [2] \\ [1] \\ 0 \end{matrix} \right\rangle = \sqrt{\frac{1}{n}}\,|1\rangle$$

$$\left| \begin{matrix} [3] \\ [2] \\ [1] \\ \begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array} \end{matrix} \right\rangle = \sqrt{\frac{n}{2(n+2)(n-1)}}\left\{\frac{2}{n}\,|1\rangle - |2\rangle - |3\rangle\right\}$$

$$\left| \begin{matrix} [3] \\ [2] \\ [1] \\ \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array} \end{matrix} \right\rangle = \sqrt{\frac{1}{2(n-1)}}\,\{|2\rangle - |3\rangle\}$$

$$\left| \begin{matrix} [3] \\ [2] \\ \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline \end{array} \end{matrix} \right\rangle = \sqrt{\frac{4}{3(n+2)(n+4)n}}\left\{|1\rangle + |2\rangle + |3\rangle - \frac{(n+2)}{2}\,(|4\rangle + |5\rangle + |6\rangle)\right\}$$

$$\left| \begin{matrix} [3] \\ [2] \\ \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array} \end{matrix} \right\rangle = \sqrt{\frac{1}{6(n-1)(n-2)n}}\{2\,|1\rangle - |2\rangle - |3\rangle - (n-1)(|4\rangle + |5\rangle + |6\rangle)\}$$

$$\left| \begin{matrix} [3] \\ [2] \\ \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} \end{array} \end{matrix} \right\rangle = \sqrt{\frac{1}{2(n-1)(n-2)n}}\,\{|2\rangle - |3\rangle - (n-1)\,(|4\rangle - |5\rangle)\} \tag{4.6c}$$

$$\left| \begin{matrix} [3] \\ \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline \end{array} \end{matrix} \right\rangle = \sqrt{\frac{1}{4(n+1)(n+4)(n+6)}}\,\{2\,(|1\rangle + |2\rangle + |3\rangle$$

$$+\ |4\rangle + |5\rangle + |6\rangle) - (n+4)(|7\rangle + |8\rangle + |9\rangle + |10\rangle)\}$$

$$\left|\begin{array}{c}[3]\\ \begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4\\\cline{1-1}\end{array}\end{array}\right\rangle=\sqrt{\frac{1}{12n(n-2)(n+1)}}\,\{2\,(|1\rangle+|2\rangle+|3\rangle)$$

$$-2\,(|4\rangle+|5\rangle+|6\rangle)-n\,(|7\rangle+|8\rangle+|9\rangle)+3n\,|10\rangle\}$$

$$\left|\begin{array}{c}[3]\\ \begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3\\\cline{1-1}\end{array}\end{array}\right\rangle=\sqrt{\frac{1}{6n(n-2)(n+1)}}\{2\,|1\rangle-|2\rangle-|3\rangle$$

$$+\,|4\rangle+|5\rangle-2\,|6\rangle-n(|7\rangle+|8\rangle-2n\,|9\rangle)\}$$

$$\left|\begin{array}{c}[3]\\ \begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2\\\cline{1-1}\end{array}\end{array}\right\rangle=\sqrt{\frac{1}{2n(n-2)(n+1)}}\,\{|2\rangle-|3\rangle+|4\rangle-|5\rangle-2\,|6\rangle-n\,(|7\rangle-|8\rangle)\}\,.$$

The matrices of g_4 and e_4 are

$$g_4=$$

$$\begin{pmatrix}
1&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0\\
0&0&0&\frac{n-2}{n(n+4)}&0&0&\sqrt{\frac{3(n+2)(n+1)(n+6)}{4(n+4)^2n}}&\sqrt{\frac{(n^2-4)(n+1)}{4(n+4)n^2}}&0&0\\
0&0&0&0&\frac{1}{n}&0&0&0&\sqrt{\frac{n^2-1}{n^2}}&0\\
0&0&0&0&0&\frac{1}{n}&0&0&0&\sqrt{\frac{n^2-1}{n^2}}\\
0&0&0&\sqrt{\frac{3(n+6)(n+2)(n+1)}{4n(n+4)^2}}&0&0&\frac{n-2}{4(n+4)}&-\sqrt{\frac{3(n+6)(n-2)}{4^2(n+4)n}}&0&0\\
0&0&0&\sqrt{\frac{(n^2-4)(n+1)}{4n^2(n+4)}}&0&0&-\sqrt{\frac{3(n+6)(n-2)}{4^2(n+4)n}}&-\frac{n-2}{4n}&0&0\\
0&0&0&0&\sqrt{\frac{n^2-1}{n^2}}&0&0&0&-\frac{1}{n}&0\\
0&0&0&0&0&\sqrt{\frac{n^2-1}{n^2}}&0&0&0&-\frac{1}{n}
\end{pmatrix}$$

(4.6*d*)

$$e_4=\begin{pmatrix}
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&\frac{3(n+2)}{n(n+4)}&0&0&\sqrt{\frac{3(n+2)(n+1)(n+6)}{4n(n+4)^2}}&-\sqrt{\frac{3^2(n^2-4)(n+1)}{4n^2(n+4)}}&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&\sqrt{\frac{3(n+1)(n+2)(n+6)}{4n(n+4)^2}}&0&0&\frac{(n+1)(n+6)}{4(n+4)}&-\sqrt{\frac{3(n+6)(n-2)(n+1)^2}{4^2n(n+4)}}&0&0\\
0&0&0&-\sqrt{\frac{3^2(n+1)(n^2-4)}{4n^2(n+4)}}&0&0&-\sqrt{\frac{3(n+1)^2(n+6)(n-2)}{4^2n(n+4)}}&\frac{3(n+1)(n-2)}{4n}&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0
\end{pmatrix}.$$

7. $f=5$, $[\lambda]=[1^3]$ with dim = 10. The uncoupled normal ordered basis vectors are

$$|1\rangle=e_1\left|(12),\begin{array}{|c|}\hline 3\\\hline 4\\\hline 5\\\hline\end{array}\right\rangle\qquad |6\rangle=g_2g_3g_1g_2\,|1\rangle$$

$$|2\rangle=g_2\,|1\rangle\qquad |7\rangle=g_4g_3g_2\,|1\rangle$$

$$|3\rangle = g_1 g_2 \,|1\rangle \qquad |8\rangle = g_4 g_3 g_1 g_2 \,|1\rangle \tag{4.7a}$$

$$|4\rangle = g_3 g_2 \,|1\rangle \qquad |9\rangle = g_2 g_4 g_3 g_1 g_2 \,|1\rangle$$

$$|5\rangle = g_3 g_1 g_2 \,|1\rangle \qquad |10\rangle = g_3 g_2 g_4 g_3 g_1 g_2 \,|1\rangle \,.$$

The norm matrix is

$$\begin{pmatrix} n & 1 & 1 & -1 & -1 & 0 & 1 & 1 & 0 & 0 \\ 1 & n & 1 & 1 & 0 & -1 & -1 & 0 & 1 & 0 \\ 1 & 1 & n & 0 & 1 & 1 & 0 & -1 & -1 & 0 \\ -1 & 1 & 0 & n & 1 & -1 & 1 & 0 & 0 & 1 \\ -1 & 0 & 1 & 1 & n & 1 & 0 & 1 & 0 & -1 \\ 0 & -1 & 1 & -1 & 1 & n & 0 & 0 & 1 & 1 \\ 1 & -1 & 0 & 1 & 0 & 0 & n & 1 & -1 & 1 \\ 1 & 0 & -1 & 0 & 1 & 0 & 1 & n & 1 & -1 \\ 0 & 1 & -1 & 0 & 0 & 1 & -1 & 1 & n & 1 \\ 0 & 0 & 0 & 1 & -1 & 1 & 1 & -1 & 1 & n \end{pmatrix} . \tag{4.7b}$$

The orthonormal basis vectors are

$$\left| \begin{array}{c} [3] \\ [1^2] \\ [1] \\ 0 \end{array} \right\rangle = \sqrt{\frac{1}{n}}\,|1\rangle$$

$$\left| \begin{array}{c} [3] \\ [1^2] \\ [1] \\ \begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array} \end{array} \right\rangle = \sqrt{\frac{n}{2(n+2)(n-1)}} \left\{ \frac{2}{n}\,|1\rangle - |2\rangle - |3\rangle \right\}$$

$$\left| \begin{array}{c} [3] \\ [1^2] \\ [1] \\ \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array} \end{array} \right\rangle = \sqrt{\frac{1}{2(n-1)}}\,\{|2\rangle - |3\rangle\}$$

$$\left| \begin{array}{c} [3] \\ [1^2] \\ \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{|c}{} \\ \cline{1-1} \end{array} \end{array} \right\rangle = \sqrt{\frac{2}{(n-1)(n^2-4)}} \left\{ |1\rangle - \frac{1}{2}\,(|2\rangle - |3\rangle) + \frac{(n-1)}{2}\,(|4\rangle + |5\rangle) \right\}$$

$$\left| \begin{array}{c} [3] \\ [1^2] \\ \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{|c}{} \\ \cline{1-1} \end{array} \end{array} \right\rangle = \sqrt{\frac{1}{6(n-1)(n^2-4)}}\,\{3\,(|2\rangle - |3\rangle) + (n-1)\,(2\,|6\rangle - |4\rangle + |5\rangle)\}$$

$$\left|\begin{array}{c}[3]\\ [2]\\ [1]\\ \boxed{1}\\ \boxed{2}\\ \boxed{3}\end{array}\right\rangle = \sqrt{\frac{1}{3(n-2)}}\,\{|4\rangle - |5\rangle + |6\rangle\} \tag{4.7c}$$

$$\left|\begin{array}{l}[3]\\ \boxed{1}\boxed{2}\\ \boxed{3}\\ \boxed{4}\end{array}\right\rangle = \sqrt{\frac{1}{2(n-3)(n^2-4)}}\{2\,|1\rangle - |2\rangle - |3\rangle$$

$$+\,|4\rangle + |5\rangle - (n-2)(|7\rangle + |8\rangle)\}$$

$$\left|\begin{array}{l}[3]\\ \boxed{1}\boxed{3}\\ \boxed{2}\\ \boxed{4}\end{array}\right\rangle = \sqrt{\frac{1}{6(n-3)(n^2-4)}}\{3(|2\rangle - |3\rangle) - |4\rangle + |5\rangle$$

$$+2\,|6\rangle + (n-2)(|7\rangle - |8\rangle - 2\,|9\rangle)\}$$

$$\left|\begin{array}{l}[3]\\ \boxed{1}\boxed{4}\\ \boxed{2}\\ \boxed{3}\end{array}\right\rangle = \sqrt{\frac{1}{12(n-3)(n^2-4)}}\{4(|4\rangle - |5\rangle + |6\rangle)$$

$$-(n-2)(|7\rangle - |8\rangle + |9\rangle) - 3(n-2)\,|10\rangle\}$$

$$\left|\begin{array}{c}[3]\\ \boxed{1}\\ \boxed{2}\\ \boxed{3}\\ \boxed{4}\end{array}\right\rangle = \sqrt{\frac{1}{4(n-3)}}\{|7\rangle - |8\rangle + |9\rangle - |10\rangle\}.$$

The matrices of g_4 and e_4 are

$$g_4 = \begin{pmatrix}
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & \frac{1}{n-2} & 0 & 0 & -\sqrt{\frac{(n-1)(n-3)}{(n-2)^2}} & 0 & 0 & 0\\
0 & 0 & 0 & 0 & \frac{1}{n-2} & 0 & 0 & -\sqrt{\frac{(n-1)(n-3)}{(n-2)^2}} & 0 & 0\\
0 & 0 & 0 & 0 & 0 & \frac{1}{n-2} & 0 & 0 & -\sqrt{\frac{(n-3)(n+2)}{4(n-2)^2}} & \sqrt{\frac{3(n-3)}{4(n-2)}}\\
0 & 0 & 0 & -\sqrt{\frac{(n-1)(n-3)}{(n-2)^2}} & 0 & 0 & -\frac{1}{n-2} & 0 & 0 & 0\\
0 & 0 & 0 & 0 & -\sqrt{\frac{(n-1)(n-3)}{(n-2)^2}} & 0 & 0 & -\frac{1}{n-2} & 0 & 0\\
0 & 0 & 0 & 0 & 0 & -\sqrt{\frac{(n-3)(n+2)}{4(n-2)^2}} & 0 & 0 & \frac{3n-10}{4(n-2)} & \sqrt{\frac{3(n+2)}{4^2(n-2)}}\\
0 & 0 & 0 & 0 & 0 & \sqrt{\frac{3(n-3)}{4(n-2)}} & 0 & 0 & \sqrt{\frac{3(n+2)}{4^2(n-2)}} & \frac{1}{4}
\end{pmatrix} \tag{4.7d}$$

$$e_4 = \begin{pmatrix} 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&\frac{3}{n-2}&0&0&-\sqrt{\frac{3^2(n-3)(n+2)}{4(n-2)^2}}&-\sqrt{\frac{3(n-3)}{4(n-2)}}\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&-\sqrt{\frac{3^2(n-3)(n+2)}{4(n-2)^2}}&0&0&\frac{3(n-3)(n+2)}{4(n-2)}&\sqrt{\frac{3(n+2)(n-3)^2}{4^2(n-2)}}\\ 0&0&0&0&0&-\sqrt{\frac{3(n-3)}{4(n-2)}}&0&0&\sqrt{\frac{3(n+2)(n-3)^2}{4^2(n-2)}}&\frac{n-3}{4} \end{pmatrix}.$$

From the above matrix representations of $D_f(n)$ generators, one can check that the representations are always faithful with n-continuation, except for $n \in \{f-2,\ f-3,\ \ldots,\ 1,\ 0\}$ or for $-n \in \{f-1,\ f-2,\ \ldots,\ 1,\ 0\}$ when n is negative. At these integer n values, the representations will become either unfaithful or indefinite, because in the latter case the denominators of some matrix elements of g_i and e_i will become zero. This is consistent with the conclusion made by Wenzl [2], and independently by Hanlon and Wales [10] that the representations of $D_f(n)$ will no longer be semisimple when $n \geqslant f-1$. Whenever $n \geqslant f-1$ and $-n > f-1$ for negative n, our results apply to any other permitted n values as well. It can also be verified that the above matrix representations of $D_f(n)$ are symmetric, and $\{g_i;\ i = 1,\ 2,\ \ldots,\ f-1\}$ are orthogonal.

In order to discuss the algebra $B_f(G)$ [14], where $G = O(n)$ or $Sp(2m)$, we need a special class of Young diagram, the so-called n-permissible Young diagrams defined in [2]. A Young diagram $[\lambda]$ is said to be n-permissible if $P_\mu(n) \neq 0$ for all subdiagrams $[\mu] \leqslant [\lambda]$, where $P_\mu(n)$ is the dimension of $O(n)$ for the irrep $[\mu]$, of which the formula is given by [12], and a Young diagram $[\mu]$ is a subdiagram of the Young diagram $[\lambda]$, denoted by $[\mu] \leqslant [\lambda]$, if $[\mu]$ can be obtained from $[\lambda]$ by taking away appropriate boxes.

The relation between $D_f(n)$ and $B_f(O(n))$ or $B_f(Sp(2m))$ was discussed in [2], in which the following corollary is of importance:

(i) If $n \in \mathbb{C}$ is not an integer, all Young diagrams are n-permissible. In this case $D_f(n) \simeq B_f(n)$ and its decomposition into full matrix rings is the same as those for $D_f(n)$.

(ii) If n is a non-zero integer, a Young diagram $[\lambda]$ is n-permissible if and only if:

   (a) Its first 2 columns contain at most n boxes for n positive.
   (b) It contains at most m columns for $n = -2m$ a negative even integer.
   (c) Its first 2 rows contain at most $2-n$ boxes for n odd and negative.

(iii) If n is a positive integer, $B_f(n) \simeq B_f(O(n))$. If n is negative and odd, $B_f(n) \simeq B_f(O(2-n))$. For $n = -2m < 0$, $B_f(n)$ is isomorphic to $B_f(Sp(2m))$.

It was proved in [2] that the generators $\{\tilde{g}_i, \tilde{e}_i\}$ of $B_f(2m)$ are compactible with the relations for $\{-g_i, -e_i\}$ of $D_f(x)$ with $x = -2m$. Hence, $g_i \mapsto -\tilde{g}_i$, $e_i \mapsto -\tilde{e}_i$ define a representation of $D_f(-2m)$, of which the image is $B_f(Sp(2m))$. Thus, making the replacements $g_i \to -\tilde{g}_i$, $e_i \to -\tilde{e}_i$, and $n \to -2m$ in the above representations of $D_f(n)$, we can obtain the matrix representations of $B_f(Sp(2m))$. In this case, an irrep $[\lambda]$ of $D_f(-2m)$ is the irrep $[\tilde{\lambda}]$ of $B_f(Sp(2m))$, where $[\tilde{\lambda}]$ is the Young diagram conjugate to $[\lambda]$.

5. Concluding remarks

In this paper, the irreducible matrix representations of Brauer algebras $D_f(n)$ are constructed by using the induced representation and the linear equation method. Some matrix representations of $D_f(n)$ for $f \leqslant 5$ are presented. Higher-dimensional representations of $D_f(n)$ can also be derived by using this method. The results are lengthy, and not presented here. As with the Schur–Weyl duality relation between S_f and $GL(n)$, the results may be useful in studying the representation theory of $O(n)$ and $Sp(2m)$ in concerning to the coupling and recoupling problems of these representations, which are now under our consideration.

This technique can also be extended to the Birman–Wenzl algebra $C_f(q, r)$ case by using the results of Hecke algebra representations [6–8]. Work in this direction is in progress.

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